

On the Eventual Periodicity of

$$x_{n+1} = \max \left\{ \frac{1}{x_n}, \frac{A_n}{x_{n-1}} \right\}$$

With a Period-Five Parameter

By

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Abstract

In this paper, we will study the max difference equation in the title when $\{A_n\}$ is a period-five sequence of real numbers greater than one. We will prove that every solution is eventually periodic with period twenty.

Key Words: asymptotic behavior, periodic solutions.

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1 Introduction

In this paper we consider the nonautonomous equation

$$x_{n+1} = \max \left\{ \frac{1}{x_n}, \frac{A_n}{x_{n-1}} \right\}, \quad n = 0, 1, \dots \quad (1.1)$$

where the initial conditions, x_{-1} and x_0 , are arbitrary positive numbers, $\{A_n\}$ is a periodic sequence with period five such that $A_n \in (1, \infty)$ for all $n \geq 0$. We will refer to these equations as max equations.

In [1], Eq.(1.1) is investigated when $\{A_n\}$ is a constant sequence with $A_n \equiv A \in (0, \infty)$. In this paper, it is proved that if $A \in (1, \infty)$, then every positive solution is eventually periodic with period four.

In [3], Eq.(1.1) is studied when $\{A_n\}$ is a periodic sequence of positive real numbers with period two. Let $A_n = A_0$ when n is even and $A_n = A_1$ when n is odd. Results include the fact that if $A_0 A_1 \in (1, \infty)$, then every positive solution is eventually periodic with period four.

In [2] and [5], Eq.(1.1) is studied when $\{A_n\}$ is a periodic sequence of positive real numbers with period three. Among the results obtained was the fact that if $A_n \in (1, \infty)$, then every positive solution is eventually periodic with period four.

In [4], Eq.(1.1) is studied when $\{A_n\}$ is a periodic sequence of positive real numbers with period four. Among the results obtained was the fact that if $A_n \in (1, \infty)$, then every positive solution is eventually periodic with period eight.

Note that in [6], the results of [1],[2],[3] and [5] are summarized within the context of other studies involving max equations of similar piecewise-defined equations. Also, open problems and conjectures on the behavior of solutions of the max equation are posed for some of the many unsolved cases which remain after the results presented here.

2 Preliminaries

We now make two definitions regarding Eq(1.1).

A **right semicycle** is a string of terms x_l, \dots, x_m with $l \geq 1, m \leq \infty$ such that

$$x_n = \frac{A_{n-1}}{x_{n-2}}$$

for $n = l, \dots, m$. Furthermore, if $l > 1$,

$$x_{l-1} = \frac{1}{x_{l-2}}$$

and if $m < \infty$,

$$x_{m+1} = \frac{1}{x_m}.$$

A **left semicycle** is a string of terms x_l, \dots, x_m with $l \geq 1, m \leq \infty$ such that

$$x_n = \frac{1}{x_{n-1}}$$

for $n = l, \dots, m$. Furthermore, if $l > 1$,

$$x_{l-1} = \frac{A_{l-2}}{x_{l-3}}$$

and if $m < \infty$,

$$x_{m+1} = \frac{A_m}{x_{m-1}}.$$

This equation was studied utilizing a program specially designed by the second author to simultaneously plot the trajectory of solutions and the curves that represent the conditions expressed in the lemmas of this manuscript. For example, it would draw curves like

$$xy = 1$$

$$y = \frac{x}{A_0}$$

and

$$y = \frac{A_2 A_1}{A_0} x.$$

If the point (x_{-1}, x_0) in the trajectory of the solution is located on the first curve and in between the latter two, then we would know that x_1 is the first term in a right semicycle with at least three terms (this fact will be detailed in the first lemma of the next section). We would then add more curves to determine whether the right semicycle would have a fourth term or not or to determine if a left semicycle would have one or two terms. By tweaking the initial conditions to match the curves drawn, we could ensure that we would find a solution with specific properties (such as a right semicycle of length four followed by a left semicycle of length two). We could also see that some situations would be impossible (conditions for the situation we are examining seem to contradict what would be needed for an extra case and then we would prove on paper that in fact the case we were examining was completely solvable without further subcases).

We will throughout the remainder of this manuscript consider whether or not the following conditions hold. We have numbered the equations so that the second number agrees with the parity of the index of the term we are considering. For example, Eq.(2.1) is needed when the index is congruent to $N + 1$. When the condition does not hold, the second number agrees with the parity but is not the remainder modulo five. So when Eq.(2.1) does not hold, Eq.(2.6) does hold.

$$\frac{A_{N+3}}{A_{N+1}} \geq \frac{A_N}{A_{N+2}} \tag{2.0}$$

$$\frac{A_{N+4}}{A_{N+2}} \geq \frac{A_{N+1}}{A_{N+3}} \tag{2.1}$$

$$\frac{A_N}{A_{N+3}} \geq \frac{A_{N+2}}{A_{N+4}} \tag{2.2}$$

$$\frac{A_{N+1}}{A_{N+4}} \geq \frac{A_{N+3}}{A_N} \tag{2.3}$$

$$\frac{A_{N+2}}{A_N} \geq \frac{A_{N+4}}{A_{N+1}} \tag{2.4}$$

$$\frac{A_{N+3}}{A_{N+1}} < \frac{A_N}{A_{N+2}} \tag{2.5}$$

$$\frac{A_{N+4}}{A_{N+2}} < \frac{A_{N+1}}{A_{N+3}} \tag{2.6}$$

$$\frac{A_N}{A_{N+3}} < \frac{A_{N+2}}{A_{N+4}} \tag{2.7}$$

$$\frac{A_{N+1}}{A_{N+4}} < \frac{A_{N+3}}{A_N} \tag{2.8}$$

and

$$\frac{A_{N+2}}{A_N} < \frac{A_{N+4}}{A_{N+1}}. \tag{2.9}$$

Consider the following equation

$$x_{n+1} = \frac{A_n}{x_{n-1}}$$

where $\{A_n\}$ is a periodic sequence with period five such that $A_n \in (1, \infty)$ for all $n \geq 0$.

We will prove that every solution of Eq.(1.1) is periodic with period twenty by showing that solutions either eventually directly repeat after twenty terms or they eventually satisfy Eq.(2) for which all nontrivial solutions are periodic with period twenty. We will do this by showing that the solution eventually reaches a point where it is periodic or we can repeatedly find a right semicycle of longer length until we get a right semicycle of length at least twenty. Some cases are trivial, for example, a right semicycle with at least eight terms has at least nine terms already. Other cases are harder. Essentially, each case (of showing that a right semicycle of a given length leads either to a periodic solution or to a longer right semicycle) is similar. For economy of exposition, we will only show one case. However, any reader interested in seeing more cases can write to the corresponding author for a longer version of this paper with all cases shown. We will show that every right semicycle has at least two terms and that we can find another right semicycle with at least three terms and omit the proofs of the other cases.

The following lemma was proven in [4] when $\{A_n\}$ is periodic with period four. The result holds when $\{A_n\}$ is periodic with period five and the proof will be omitted.

Lemma 2.1 *For Eq.(1.1), every left semicycle has length one or two. Furthermore, if x_N is a term in a left semicycle, then $x_N x_{N-1} = 1$.*

Throughout, when we assume that x_N is a term in a left semicycle, we will consistently make use of the fact mentioned in the previous lemma, that $x_N x_{N-1} = 1$ when performing calculations.

The proof of the following lemma follows by simple induction and the fact that Eq.(2) is periodic with period twenty.

Lemma 2.2 *Let $\{x_n\}$ be a solution of Eq.(1.1) such that x_N is the last term in a left semicycle. If the next right semicycle has length at least twenty, then $\{x_n\}$ is eventually periodic. In fact, $x_n = x_{n-20}$ for $n \geq N + 1$ and the right semicycle starting with x_{N+1} has an infinite number of terms.*

3 Right Semicycles of Length Two

We will now consider solutions which have right semicycles of length two.

Lemma 3.1 *For Eq.(1.1), the following are true about right semicycles. Assume N is a nonnegative integer such that x_N is the last term in a left semicycle.*

1) *Every right semicycle has at least two terms.*

2) *The right semicycle starting with x_{N+1} has exactly two terms iff*

$$\frac{A_{N+2}A_{N+1}}{A_N}x_{N-1} < x_N . \tag{3.1}$$

3) *The right semicycle starting with x_{N+1} has at least three terms iff*

$$\frac{x_{N-1}}{A_N} \leq x_N \leq \frac{A_{N+2}A_{N+1}x_{N-1}}{A_N} . \tag{3.2}$$

Lemma 3.2 *Assume N is a nonnegative integer such that x_N is the last term in a left semicycle. The following are true.*

1) x_{N+1} is the first term of a right semicycle of length two preceding a left semicycle of length one iff

$$x_N > \max \left\{ \frac{A_{N+1}}{A_{N+3}}, \frac{A_{N+2}}{A_N} \right\} A_{N+1} x_{N-1} . \quad (3.3)$$

2) x_{N+1} is the first term of a right semicycle of length two preceding a left semicycle of length two iff Eq.(2.5) holds and

$$\frac{A_{N+1}A_{N+2}}{A_N} x_{N-1} < x_N < \frac{A_{N+1}^2}{A_{N+3}} x_{N-1} . \quad (3.4)$$

Before stating and proving the main result of this section, let us explain the convention that we will use. If we say that Eq.(3.4) holds, we mean that it holds for the index N . If we say that Eq.(3.4) holds between x_{N+3} and x_{N+4} , we mean that the equation holds if the index N is replaced with the index $N+4$, i.e.

$$\frac{A_{N+5}A_{N+6}}{A_{N+4}} x_{N+3} < x_{N+4} < \frac{A_{N+5}^2}{A_{N+7}} x_{N+3} .$$

In other words,

$$\frac{A_N A_{N+1}}{A_{N+4}} x_{N+3} < x_{N+4} < \frac{A_N^2}{A_{N+2}} x_{N+3} .$$

We will now prove that every solution has a right semicycle of length greater than two.

Lemma 3.3 *Assume N is a nonnegative integer such that x_N is the last term in a left semicycle. Then there exists an integer $M > N$ such that x_M is the last term in a left semicycle and Eq.(3.2) holds.*

Proof. First suppose that Eq.(3.4) never holds. Then every right semicycle with length two is succeeded by a left semicycle of length one. There exists a positive integer l such that

$$x_N \leq \frac{A_{N+3l+2}}{A_{N+3l}} A_{N+3l+1} A_{N+1}^{2l} x_{N-1} .$$

We have by induction that for $k = 0, \dots, l$

$$x_{N+3k-1} = A_{N+1}^k x_{N-1}$$

and

$$x_{N+3k} = \frac{x_N}{A_{N+1}^k} .$$

Note that

$$x_{N+3l} \leq \frac{A_{N+3l+2}}{A_{N+3l}} A_{N+3l+1} x_{N+3l-1} .$$

Therefore, x_{N+3l} is the (last) term in a left semicycle and Eq.(3.2) now holds and $M = N + 3l$.

We now suppose that there is an index, without loss of generality, say it is N , such that Eq.(3.4) holds (which implies that Eq.(2.5) holds). So, x_{N+4} is the last term of a left semicycle,

$$x_{N+3} = \frac{x_N}{A_{N+1}}$$

and

$$x_{N+4} = A_{N+1} x_{N-1} .$$

If

$$x_{N+4} < \frac{A_N}{A_{N+4}} A_{N+1} x_{N+3}$$

then Eq.(3.2) holds between x_{N+3} and x_{N+4} . So, $M = N + 4$.

So now we assume

$$x_N < \frac{A_{N+4}}{A_N} A_{N+1} x_{N-1} \quad (3.5)$$

which is necessary for

$$x_{N+4} > \frac{A_{N+2}}{A_N} A_{N+1} x_{N-1}.$$

Eq.(3.5) implies

$$A_{N+4} > A_{N+2} \quad (3.6)$$

There are now two cases.

Case I: Eq.(3.3) holds between x_{N+3} and x_{N+4} .

By assumption, x_{N+7} is the last term in a left semicycle. If

$$x_N > \frac{A_{N+2}}{A_N^2 A_{N+3} A_{N+4}} A_{N+1}^2 x_{N-1}$$

then

$$x_{N+7} < \frac{A_{N+4}}{A_{N+2}} A_{N+3} x_{N-1}$$

and $M = N + 7$.

Otherwise, we assume

$$x_N \leq \frac{A_{N+2}}{A_N^2 A_{N+3} A_{N+4}} A_{N+1}^2 x_{N-1}$$

which requires

$$A_{N+1} > A_{N+4} A_{N+3} A_N \quad (3.7)$$

to be consistent with Eq.(3.4). If x_{N+10} is the last term in the next left semicycle, then it follows that $M = N + 10$ since

$$A_{N+2}^2 > \frac{1}{A_{N+3}^2}$$

implies that

$$\frac{A_{N+1} A_{N+2}}{A_N} > \frac{A_{N+1}}{A_{N+3}^2 A_{N+2} A_N}.$$

Therefore,

$$x_N > \frac{A_{N+1}}{A_{N+3}^2 A_{N+2} A_N} x_{N-1}$$

which gives us that

$$x_{N+10} < \frac{A_{N+2} A_{N+1}}{A_N} x_{N+9}.$$

If x_{N+10} is not the last term in the next left semicycle, then x_{N+11} is the last term.

It is easy to see that Eq.(3.1) now holds between x_{N+10} and x_{N+11} . Eq.(3.4) implies

$$x_N > \frac{A_{N+2} A_{N+1}}{A_{N+3} A_N^2} x_{N-1}$$

and so

$$x_{N+11} = \frac{A_{N+3}A_N}{A_{N+1}}x_N > \frac{A_{N+3}A_{N+2}}{A_{N+1}} \frac{A_{N+1}}{A_{N+3}A_N}x_{N-1} = \frac{A_{N+3}A_{N+2}}{A_{N+1}}x_{N+10}.$$

Therefore, we know the next right semicycle has length two. If Eq.(3.3) holds between x_{N+10} and x_{N+11} , then Eq.(3.7) implies

$$A_{N+1} > \frac{A_{N+3}A_{N+4}}{A_{N+2}^2}$$

and together with Eq.(3.4), we have

$$x_{N+14} < \frac{A_{N+1}A_N}{A_{N+4}}x_{N+13}.$$

So $M = N + 14$.

Now we assume Eq.(3.4) holds between x_{N+10} and x_{N+11} . Eq.(3.4) implies that

$$x_N > \frac{A_{N+2}A_{N+1}}{A_{N+3}^2A_N}x_{N-1}$$

which gives us

$$x_{N+15} < \frac{A_{N+2}A_{N+1}}{A_N}x_{N+14}.$$

$M = N + 15$.

Case II: Eq.(3.4) holds (between x_{N+3} and x_{N+4}).

We must have that

$$\frac{A_N A_{N+1}}{A_{N+4}}x_{N+3} < x_{N+4} < \frac{A_N^2}{A_{N+2}}x_{N+3}.$$

The latter equation implies that x_{N-1} and x_N must satisfy

$$\frac{A_{N+1}A_{N+2}}{A_N^2}A_{N+1}x_{N-1} < x_N < \frac{A_{N+4}}{A_N}A_{N+1}x_{N-1}. \quad (3.8)$$

In order for this equation to not contradict Eq.(3.4), we have that

$$A_{N+4} > A_{N+2}. \quad (3.9)$$

If

$$x_N \leq \frac{A_{N+4}A_{N+1}^2}{A_{N+3}A_N}x_{N-1}.$$

then Eq.(3.2) holds and $M = N + 8$. Otherwise, we assume

$$x_N > \frac{A_{N+4}A_{N+1}^2}{A_{N+3}A_N}x_{N-1} \quad (3.10)$$

which requires

$$A_{N+1} < A_{N+3} \quad (3.11)$$

to be consistent with Eq.(3.4) considering $A_{N+4} > A_{N+2}$.

Together with Eq.(2.5), we have

$$A_{N+2} < A_N. \quad (3.12)$$

Case IIa: Eq.(3.3) holds between x_{N+7} and x_{N+8} .

If

$$A_N A_{N+1} < A_{N+3} A_{N+2} A_{N+4},$$

then Eq.(3.2) holds between x_{N+10} and x_{N+11} and $M = N + 11$.

Now, we claim that Eq.(3.3) holds between x_{N+10} and x_{N+11} and then we will show that this implies $M = N + 14$. In order for Eq.(3.4) to hold between x_{N+10} and x_{N+11} , we would need

$$\frac{A_{N+3}}{A_{N+1}} < \frac{A_{N+2}}{A_{N+4}}.$$

This equation together with Eq.(3.11) implies that $A_{N+2} > A_{N+4}$ which contradicts Eq.(3.9).

So we have established the claim and Eq.(3.3) must hold between x_{N+10} and x_{N+11} , i.e. x_{N+14} is the last term in a left semicycle. Eq.(3.4) implies that Eq.(3.2) holds between x_{N+13} and x_{N+14} . Therefore, $M = N + 14$.

Case IIb: Eq.(3.4) holds between x_{N+7} and x_{N+8} .

We claim that $M = N + 12$. By Eq.(3.10) and Eq.(3.12), we have

$$x_N > \frac{A_{N+4}A_{N+1}^2}{A_{N+3}A_N}x_{N-1} > \frac{A_{N+4}A_{N+1}^2A_{N+2}}{A_{N+3}A_N^2}x_{N-1}$$

which implies that

$$x_{N+12} = \frac{A_{N+4}A_{N+1}}{A_N}x_{N-1} < \frac{A_{N+4}A_{N+1}}{A_N} \frac{A_{N+3}A_N^2}{A_{N+4}A_{N+1}^2A_{N+2}}x_N = \frac{A_{N+3}A_N}{A_{N+1}A_{N+2}}x_N = \frac{A_{N+3}A_{N+4}}{A_{N+2}}x_{N+11}.$$

So, $M = N + 12$ as claimed. □

As stated above, the lemmas and proofs showing that the solution is either eventually periodic (within a few terms and we can determine when it starts repeating) or find a longer right semicycle have been omitted for brevity.

Theorem 3.1 *Every solution of Eq.(1.1) is eventually periodic with (not necessarily prime) period twenty.*

4 References

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